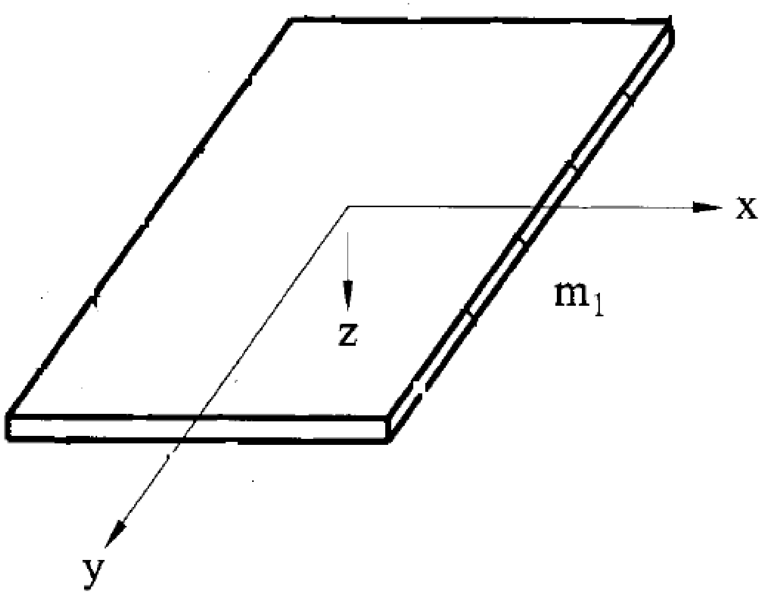
Rectangular elastic plates on Winkler soil

(Plates on Grade)

Fundamental equations

Kirchhoff theory of thin plates

$$\frac{\partial^4 w}{\partial x^4} + 2 \cdot \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$



$q = q(x, y)$ Normal load function

$w = w(x, y)$ Normal displacement function

Equation of plates on an elastic foundation

Winkler soil

$$\sigma = -k_0 \cdot w(x, y)$$

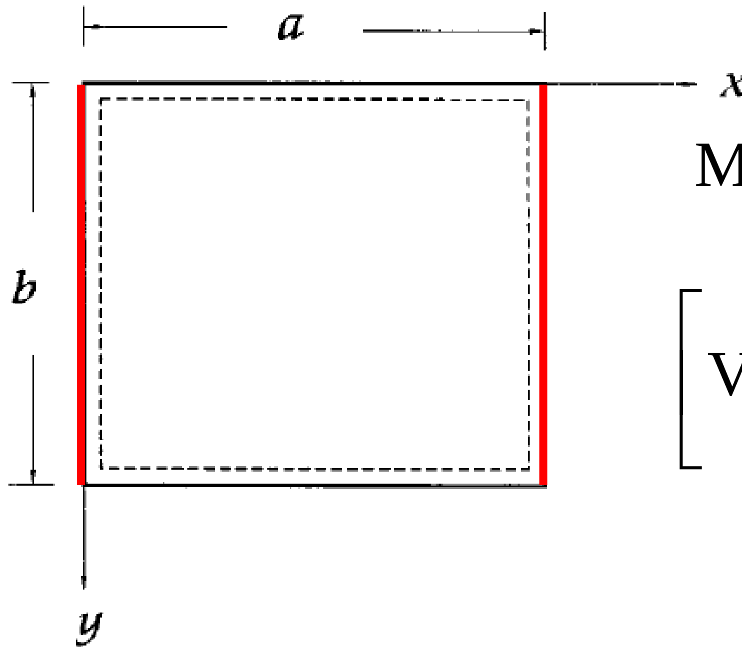
$$\frac{\partial^4 w}{\partial x^4} + 2 \cdot \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{k_0 w}{D} = \frac{q}{D}$$

Fundamental equations

Boundary conditions: free plate

$$M_{x|x=0} = 0$$

$$\left[V_x + \frac{\partial M_{xy}}{\partial y} \right]_{|x=0} = 0$$



$$M_{x|x=a} = 0$$

$$\left[V_x + \frac{\partial M_{xy}}{\partial y} \right]_{|x=a} = 0$$

$$M_x = -D \cdot \left(\frac{\partial^2 w}{\partial x^2} + \nu \cdot \frac{\partial^2 w}{\partial y^2} \right)$$

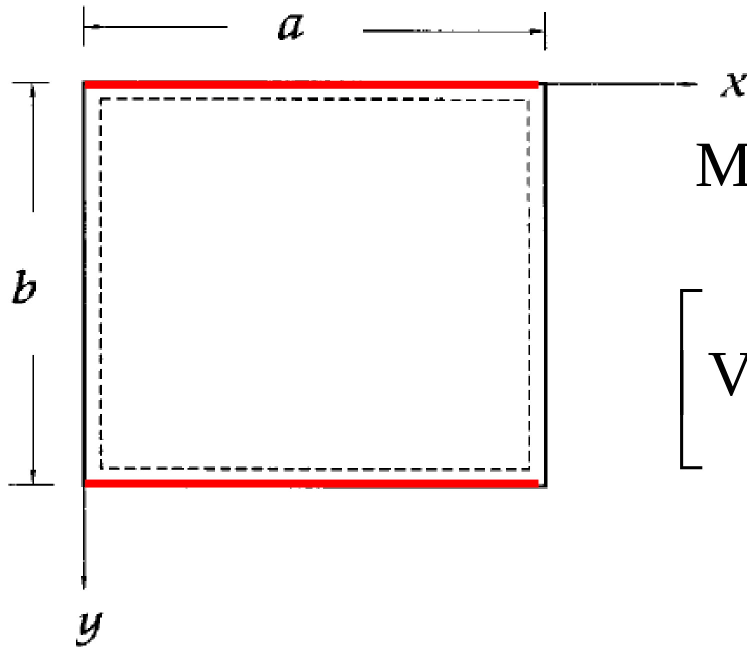
$$R_x = -D \cdot \left(\frac{\partial^3 w}{\partial x^3} - (2 - \nu) \cdot \frac{\partial^3 w}{\partial x \partial y^2} \right)$$

Fundamental equations

Boundary conditions: free plate

$$M_{y|y=0} = 0$$

$$\left[V_y + \frac{\partial M_{yx}}{\partial x}\right]_{|y=0} = 0$$



$$M_{y|y=b} = 0$$

$$\left[V_y + \frac{\partial M_{yx}}{\partial x}\right]_{|y=b} = 0$$

$$M_y = -D \cdot \left(\frac{\partial^2 w}{\partial y^2} + \nu \cdot \frac{\partial^2 w}{\partial x^2}\right)$$

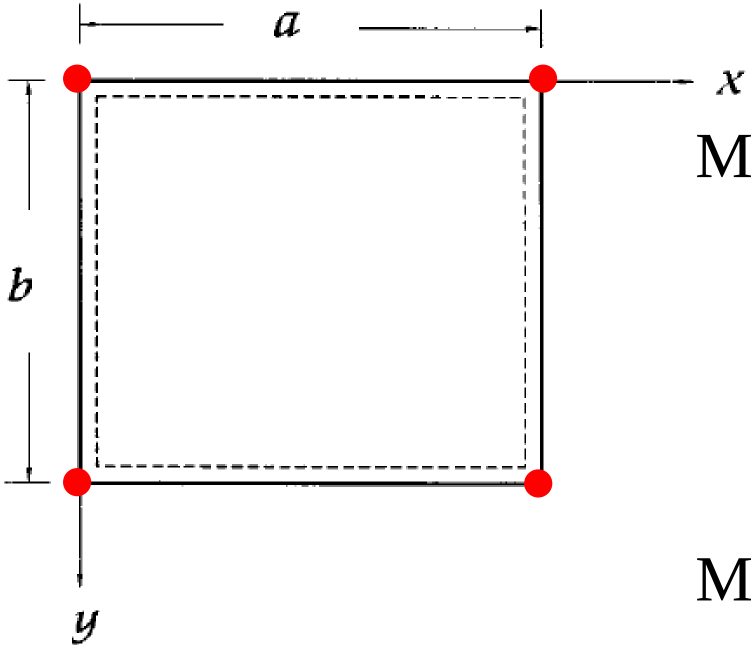
$$R_y = -D \cdot \left(\frac{\partial^3 w}{\partial y^3} - (2-\nu) \cdot \frac{\partial^3 w}{\partial x^2 \partial y}\right)$$

Fundamental equations

Boundary conditions: free plate

$$M_{xy|x=0,y=0} = 0$$

$$M_{xy|x=a,y=0} = 0$$



$$M_{xy|x=0,y=b} = 0$$

$$M_{xy|x=a,y=b} = 0$$

$$M_{xy} = M_{yx} = -(1-\nu)\cdot D\cdot\frac{\partial^2 w}{\partial x\partial y}$$

Finite difference

Central difference expressions

$$w_{M,N} = w\left(x_M, y_N\right)$$

$$\left.\frac{\partial w}{\partial x}\right|_{\substack{x=x_M \\ y=y_N}} = \frac{w_{M+1,N} - w_{M-1,N}}{2 \cdot \Delta}$$

$$\left.\frac{\partial^2 w}{\partial x^2}\right|_{\substack{x=x_M \\ y=y_N}} = \frac{w_{M+1,N} - 2 \cdot w_{M,N} + w_{M-1,N}}{\Delta^2}$$

$$\left.\frac{\partial^3 w}{\partial x^3}\right|_{\substack{x=x_M \\ y=y_N}} = \frac{w_{M+2,N} - 2 \cdot w_{M+1,N} + 2 \cdot w_{M-1,N} - w_{M-2,N}}{2 \cdot \Delta^3}$$

$$\left.\frac{\partial^4 w}{\partial x^4}\right|_{\substack{x=x_M \\ y=y_N}} = \frac{w_{M+2,N} - 4 \cdot w_{M+1,N} + 6 \cdot w_{M,N} - 4 \cdot w_{M-1,N} + w_{M-2,N}}{\Delta^4}$$

Finite difference

Central difference expressions

$$w_{M,N} = w\left(x_M, y_N\right)$$

$$\left.\frac{\partial w}{\partial y}\right|_{\substack{x=x_M \\ y=y_N}} = \frac{w_{M,N+1} - w_{M,N-1}}{2 \cdot \Delta}$$

$$\left.\frac{\partial^2 w}{\partial y^2}\right|_{\substack{x=x_M \\ y=y_N}} = \frac{w_{M,N+1} - 2 \cdot w_{M,N} + w_{M,N-1}}{\Delta^2}$$

$$\left.\frac{\partial^3 w}{\partial y^3}\right|_{\substack{x=x_M \\ y=y_N}} = \frac{w_{M,N+2} - 2 \cdot w_{M,N+1} + 2 \cdot w_{M,N-1} - w_{M,N-2}}{2 \cdot \Delta^3}$$

$$\left.\frac{\partial^4 w}{\partial y^4}\right|_{\substack{x=x_M \\ y=y_N}} = \frac{w_{M,N+2} - 4 \cdot w_{M,N+1} + 6 \cdot w_{M,N} - 4 \cdot w_{M,N-1} + w_{M,N-2}}{\Delta^4}$$

Finite difference

Central difference expressions

$$\left.\frac{\partial^2 w}{\partial x \partial y}\right|_{\substack{x=x_M \\ y=y_N}} = \left.\frac{\partial}{\partial y}\frac{\partial w}{\partial x}\right|_{\substack{x=x_M \\ y=y_N}} = \frac{w_{M+1,N+1} - w_{M+1,N-1} - w_{M-1,N+1} + w_{M-1,N-1}}{4\cdot\Delta^2}$$

$$\left.\frac{\partial^4 w}{\partial x^2 \partial y^2}\right|_{\substack{x=x_M \\ y=y_N}} = \left.\frac{\partial^2}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right|_{\substack{x=x_M \\ y=y_N}} =$$

$$= \frac{w_{M+1,N+1} - 2w_{M+1,N} + w_{M+1,N-1} - 2\cdot w_{M,N+1} + 4\cdot w_{M,N+1} - 2\cdot w_{M,N-1} + w_{M-1,N+1} - 2w_{M-1,N} + w_{M-1,N-1}}{\Delta^4}$$

$$\left.\frac{\partial^3 w}{\partial x \partial y^2}\right|_{\substack{x=x_M \\ y=y_N}} = \left.\frac{\partial}{\partial x}\frac{\partial^2 w}{\partial y^2}\right|_{\substack{x=x_M \\ y=y_N}} = \frac{w_{M+1,N+1} - w_{M-1,N+1} - 2\cdot w_{M+1,N} + 2\cdot w_{M-1,N} + w_{M+1,N-1} - w_{M-1,N-1}}{2\cdot\Delta^3}$$

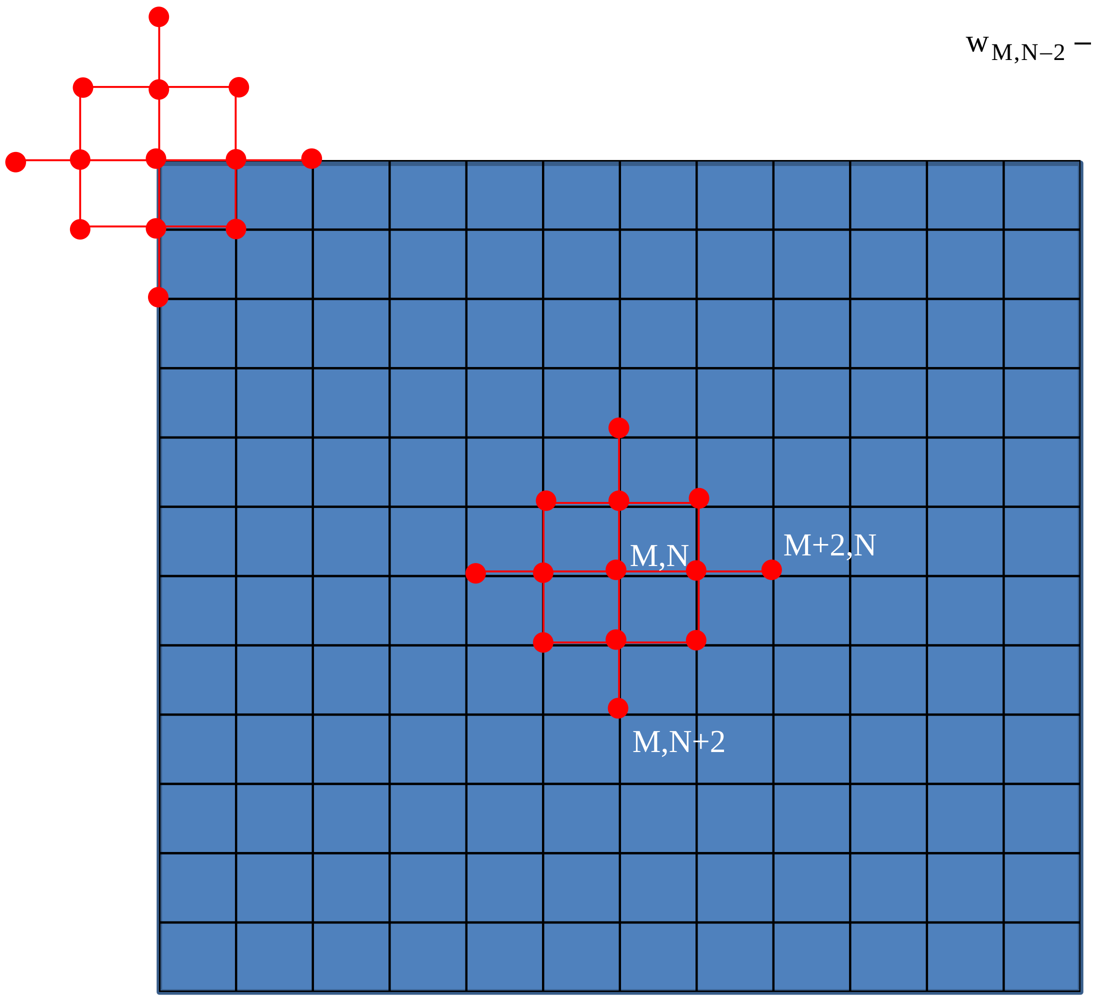
$$\left.\frac{\partial^3 w}{\partial x^2 \partial y}\right|_{\substack{x=x_M \\ y=y_N}} = \left.\frac{\partial}{\partial y}\frac{\partial^2 w}{\partial x^2}\right|_{\substack{x=x_M \\ y=y_N}} = \frac{w_{M+1,N+1} - w_{M+1,N-1} - 2\cdot w_{M,N+1} + 2\cdot w_{M,N-1} + w_{M-1,N+1} - w_{M-1,N-}}{2\Delta^3}$$

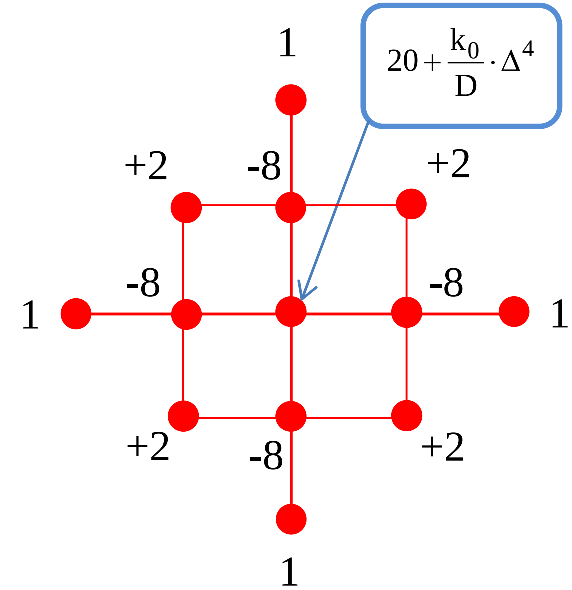
Finite difference

Central difference expressions

$$\frac{\partial^4 w}{\partial x^4} + 2\cdot\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{k_0 w}{D} = \frac{q}{D}$$

$$\begin{aligned}
& w_{M+2,N} + \\
& 2\cdot w_{M+1,N+1} - 8\cdot w_{M+1,N} + 2\cdot w_{M+1,N-1} + \\
& w_{M,N-2} - 8\cdot w_{M,N-1} + \left(20 + \frac{k_0}{D}\cdot\Delta^4\right)\cdot w_{M,N} - 8\cdot w_{M,N+1} + w_{M,N+2} + \\
& 2\cdot w_{M-1,N+1} - 8\cdot w_{M-1,N} + 2\cdot w_{M-1,N-1} + \\
& w_{M-2,N} = \\
& = \frac{q_{M,N}}{D}\cdot\Delta^4
\end{aligned}$$

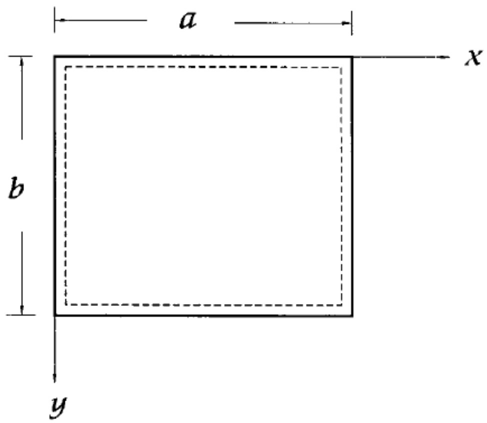




$$20 + \frac{k_0}{D}\cdot\Delta^4$$

Finite difference

Central difference expressions



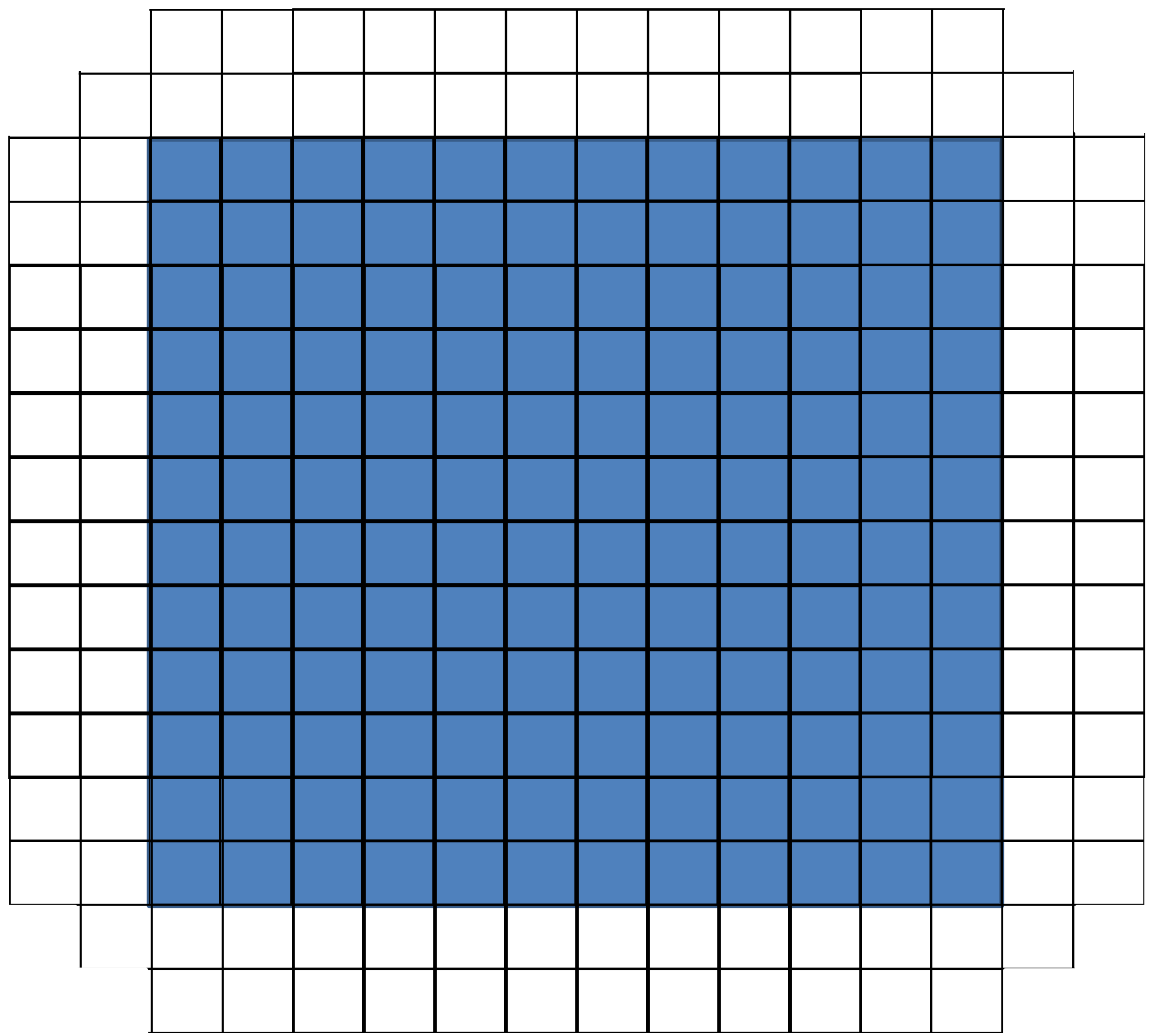
$$\bar{M} = \frac{a}{\Delta} + 1 \qquad \bar{N} = \frac{b}{\Delta} + 1$$

Total number of <u>unknowns</u>:

$$\bar{M} \cdot \bar{N} + 4 \cdot \left(\bar{M} + \bar{N} + 1 \right)$$

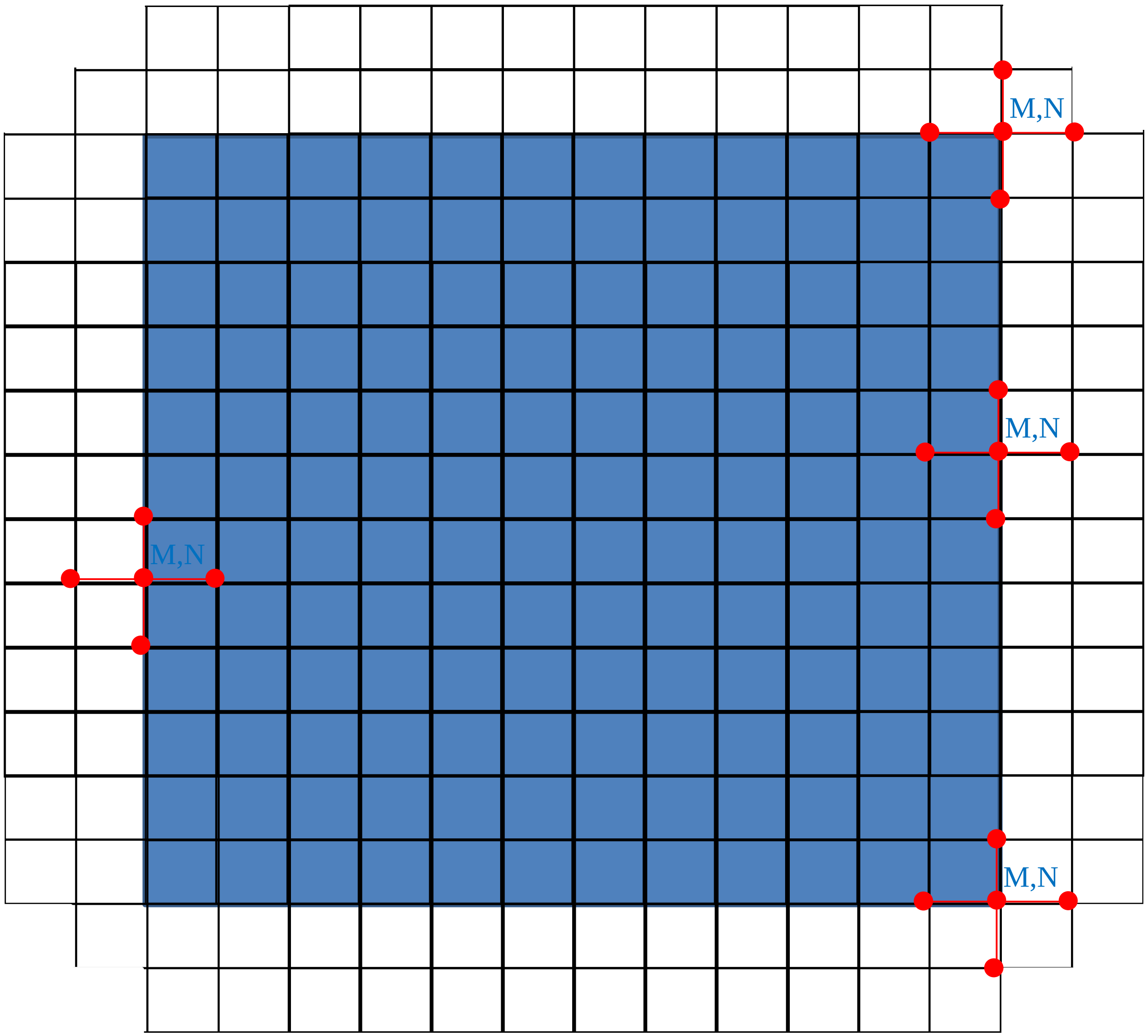
Total number of <u>equations</u>:

$$\bar{M} \cdot \bar{N}$$



Finite difference

Central difference expressions



$$-D\cdot\left(\frac{\partial^2 w}{\partial x^2}+\nu\cdot\frac{\partial^2 w}{\partial y^2}\right)\bigg|_{x=a}=0$$

$$\nu\cdot w_{M,N+1} + w_{M-1,N} - 2\cdot(1+\nu)\cdot w_{M,N} + w_{M+1,N} + \nu\cdot w_{M,N-1} = 0$$

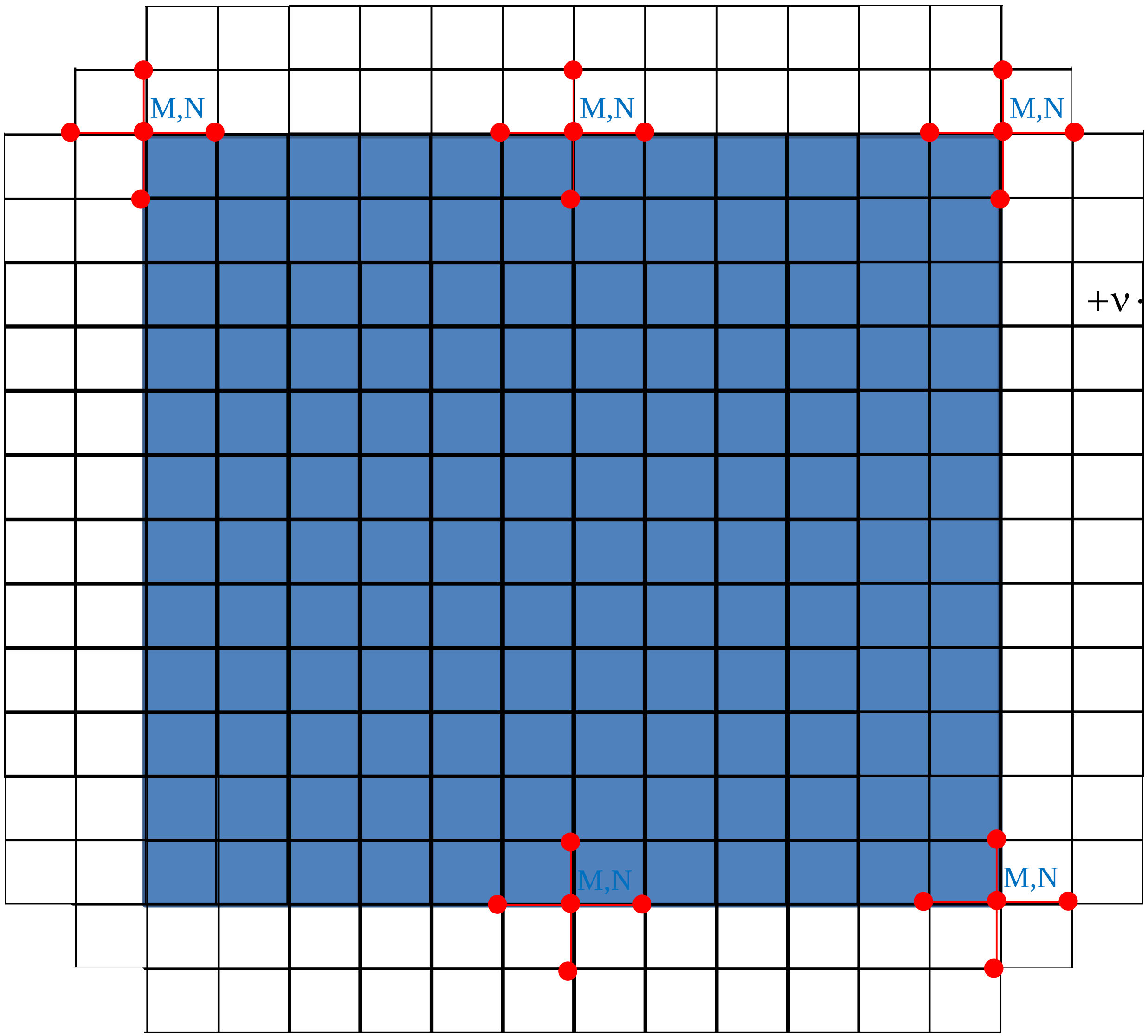
$$-D\cdot\left(\frac{\partial^2 w}{\partial x^2}+\nu\cdot\frac{\partial^2 w}{\partial y^2}\right)\bigg|_{x=0}=0$$

Total number of new equations:

$$2\cdot\bar{N}$$

Finite difference

Central difference expressions



$$-D\cdot\left(\frac{\partial^2 w}{\partial y^2}+\nu\cdot\frac{\partial^2 w}{\partial x^2}\right)\bigg|_{y=b}=0$$

$$\begin{aligned} & w_{M+1,N} \\ +\nu\cdot w_{M,N-1}-2\cdot(1+\nu)\cdot w_{M,N}+\nu\cdot & w_{M,N+1} \\ & +w_{M-1,N} \\ & =0 \end{aligned}$$

$$-D\cdot\left(\frac{\partial^2 w}{\partial y^2}+\nu\cdot\frac{\partial^2 w}{\partial x^2}\right)\bigg|_{y=0}=0$$

Total number of new equations:

$$2\cdot\bar{M}$$

Total number of equations:

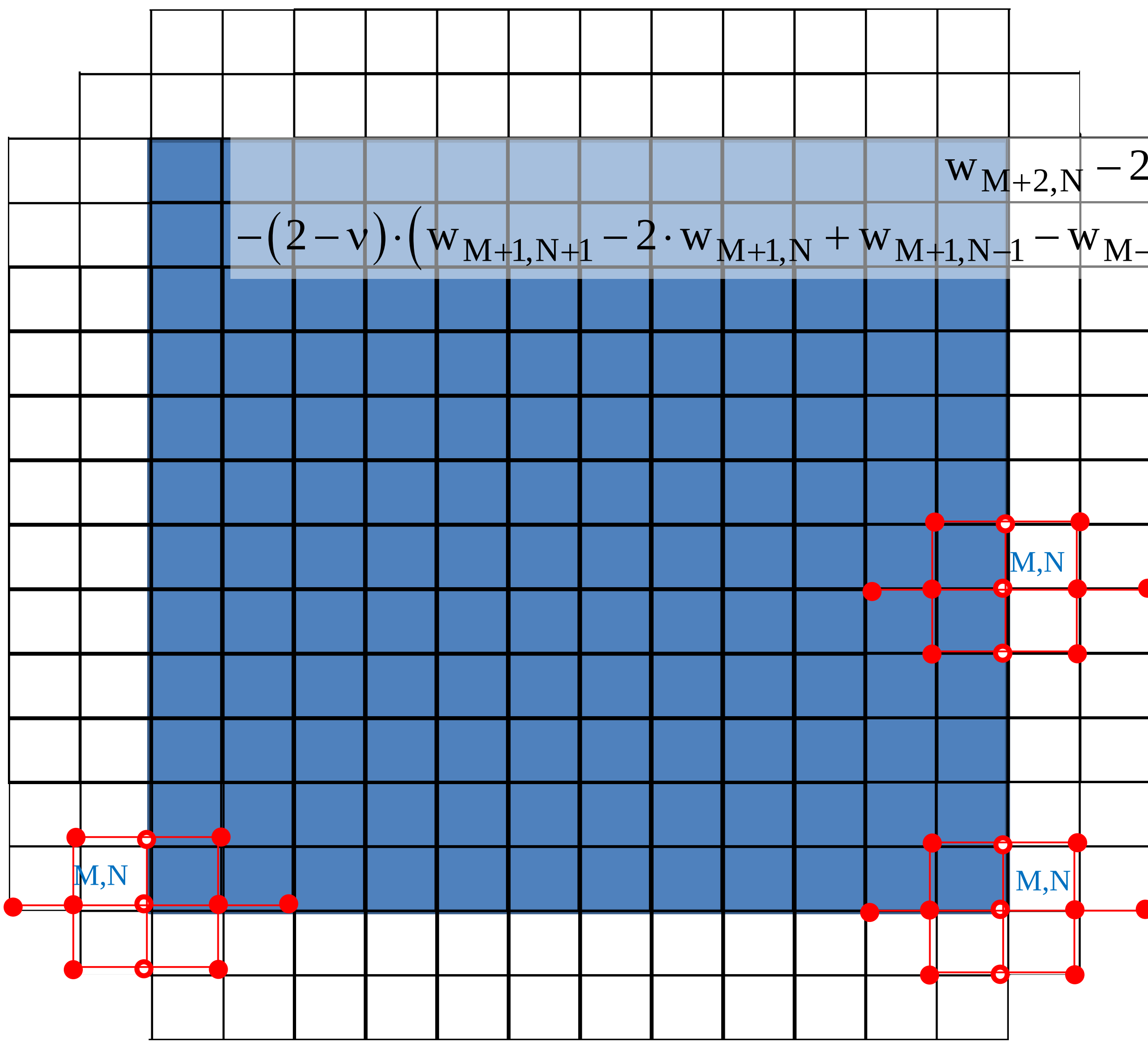
$$\bar{M}\cdot\bar{N}+2\cdot\left(\bar{M}+\bar{N}\right)$$

Finite difference

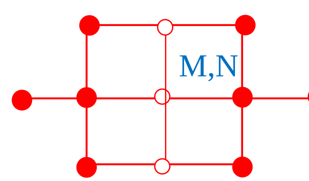
Central difference expressions

$$-D\cdot\left(\frac{\partial^3 w}{\partial x^3}-(2-\nu)\cdot\frac{\partial^3 w}{\partial x\partial y^2}\right)_{x=a}=0$$

$$w_{M+2,N}-2\cdot w_{M+1,N}+2\cdot w_{M-1,N}-w_{M-2,N}+$$
$$-(2-\nu)\cdot\left(w_{M+1,N+1}-2\cdot w_{M+1,N}+w_{M+1,N-1}-w_{M-1,N+1}+2\cdot w_{M-1,N}-w_{M-1,N-1}\right)=0$$



$$-D\cdot\left(\frac{\partial^3 w}{\partial x^3}-(2-\nu)\cdot\frac{\partial^3 w}{\partial x\partial y^2}\right)_{x=0}=0$$

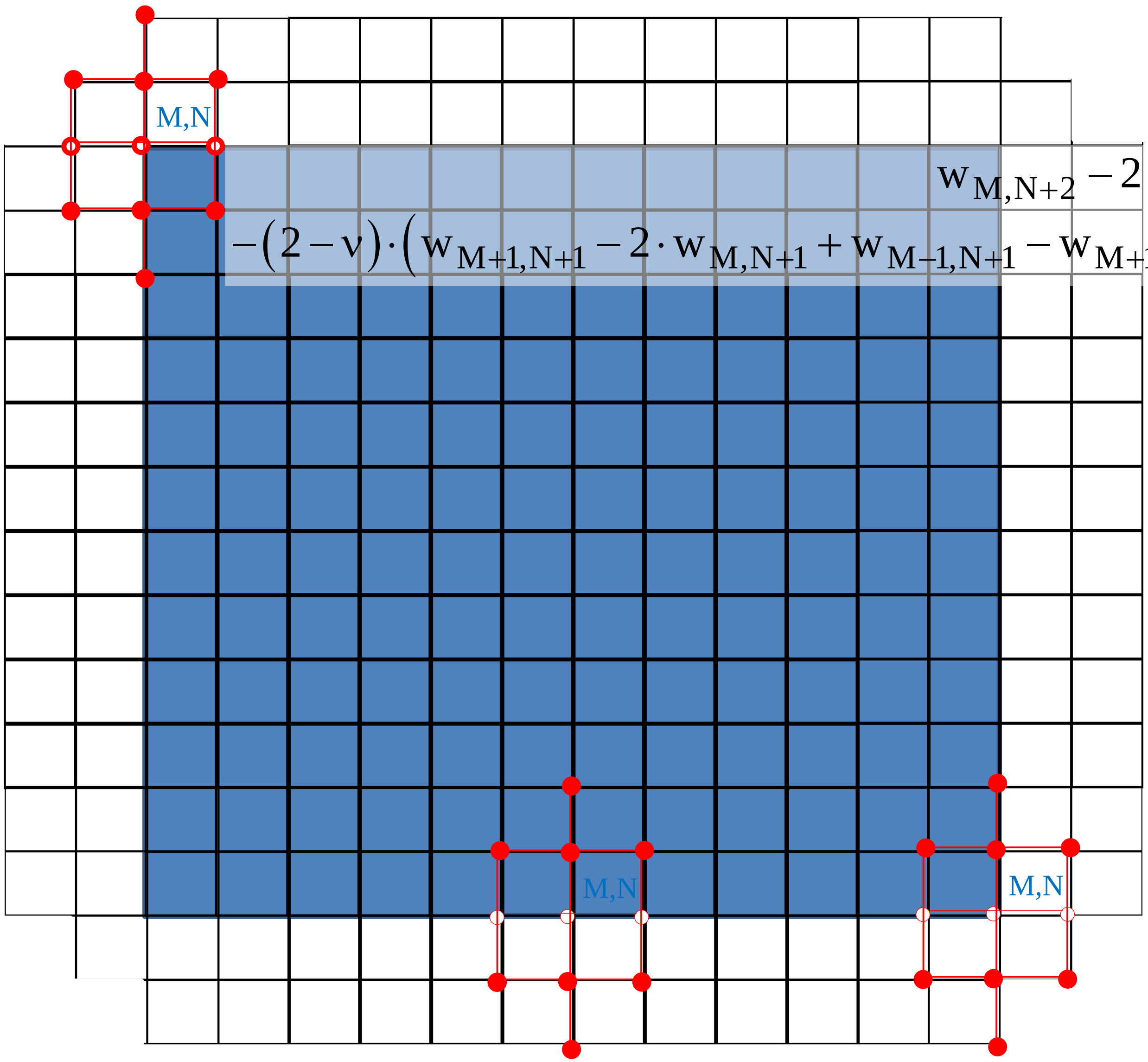


Total number of <u>new</u> <u>equations</u>:

$$2\cdot\bar{N}$$

Finite difference

Central difference expressions

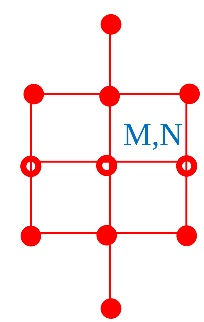


$$-D\cdot\left(\frac{\partial^3 w}{\partial y^3}-(2-\nu)\cdot\frac{\partial^3 w}{\partial x^2\partial y}\right)_{y=b}=0$$

$$w_{M,N+2}-2\cdot w_{M,N+1}+2\cdot w_{M,N-1}-w_{M,N-2}+$$

$$-(2-\nu)\cdot\left(w_{M+1,N+1}-2\cdot w_{M,N+1}+w_{M-1,N+1}-w_{M+1,N-1}+2\cdot w_{M,N-1}-w_{M-1,N-1}\right)=0$$

$$-D\cdot\left(\frac{\partial^3 w}{\partial y^3}-(2-\nu)\cdot\frac{\partial^3 w}{\partial x^2\partial y}\right)_{y=0}=0$$



Total number of <u>new</u> <u>equations</u>:

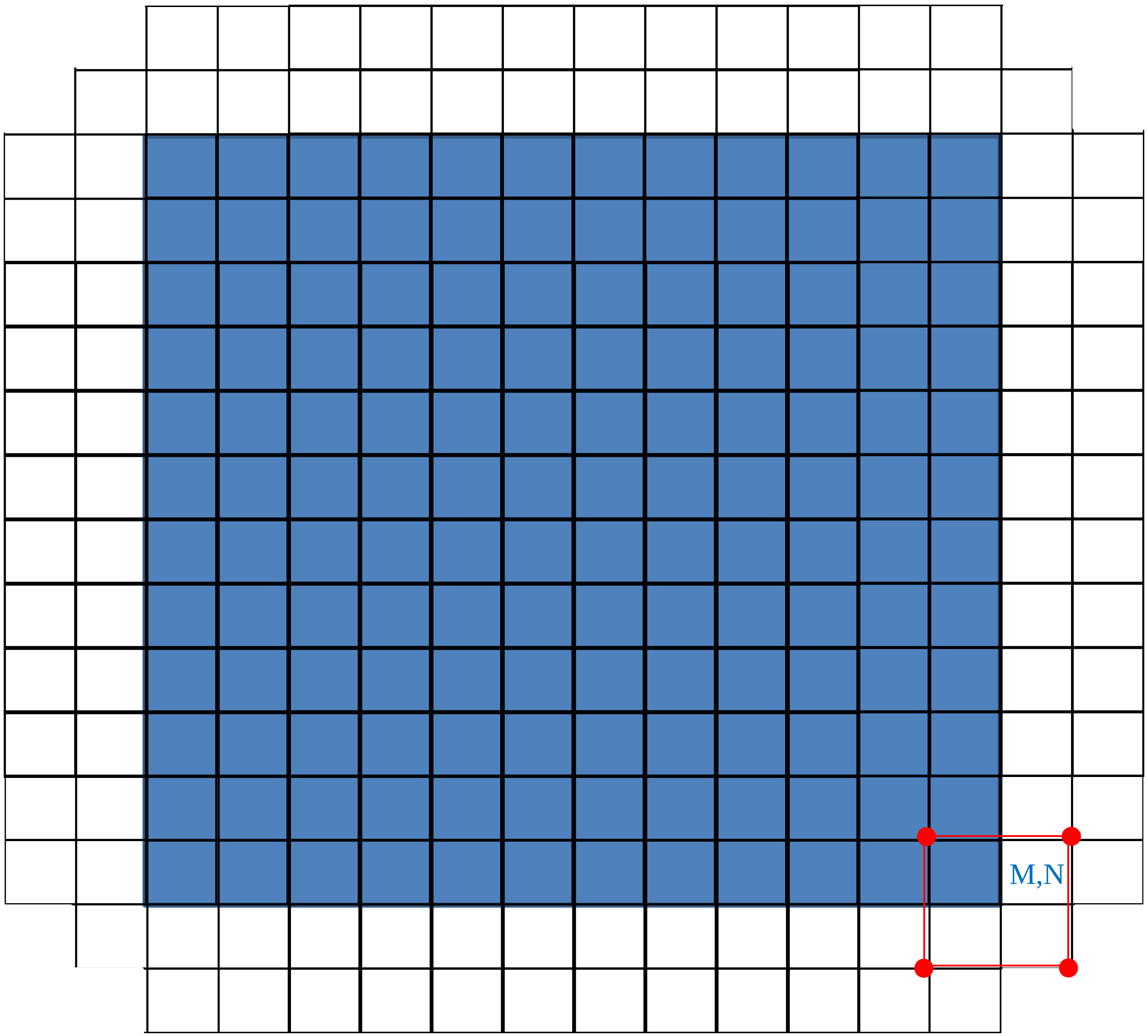
$$2\cdot\bar{M}$$

Total number of <u>equations</u>:

$$\bar{M}\cdot\bar{N}+4\cdot\left(\bar{M}+\bar{N}\right)$$

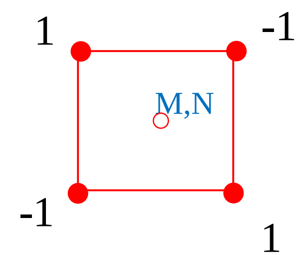
Finite difference

Central difference expressions



$$-(1-\nu)\cdot D\cdot\left.\frac{\partial^2 w}{\partial x\partial y}\right|_{\substack{x=a\\ y=b}}=0$$

$$w_{M+1,N+1}-w_{M+1,N-1}+$$
$$-w_{M-1,N+1}+w_{M-1,N-1}=0$$



Total number of <u>new</u> <u>equations</u>:
4

Total number of <u>equations</u>:
$$\bar{M}\cdot\bar{N}+4\cdot(\bar{M}+\bar{N}+1)$$