STABILITY OF STRUCTURES

with emphasis on steel ones

Short collection of notes of the course

"Stability of Steel Structures"

within the Master Course

"Design of Steel Structures"

University of Naples "Federico II", Naples (Italy)

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1. Basic Concepts

1.1 Introduction

The present chapter introduces the basic concepts about stability of structures. Making reference to simple discrete systems, it deals with the basic methods and approaches to be applied and pursuit when the study of structures stability is of concern.

Definition of equilibrium conditions is the first subject of the discussion; it is based on "quantitative" relationships involving forces acting on the system and its relevant geometric and mechanical properties, such as dimensions, stiffness and mass. Discussion about stability of structures begins just after such a definition of equilibrium conditions; it basically examines the aspect of the "quality" of the equilibrium configuration possibly achieved by the system. As intuitive, "quality" of equilibrium – for both rigid bodies and elastic or even non-linear systems - can be classified on the basis of its possible evolution after a possible (even small) perturbation on the system while resting in its equilibrium condition. Under this standpoint, basic mechanical knowledge and physical every-day intuition draw to the following three possible definitions for an equilibrium configuration of the system:

- stable equilibrium, if the system after perturbation come back to the initial equilibrium configuration, possible through small oscillations in its neighbours;
- unstable equilibrium, if a perturbation however small in intensity results in a change of the initial equilibrium configuration;
- neutral equilibrium, if the system achieves in a new equilibrium configuration, passing through other equilibrium configurations, depending on the intensity of the perturbation.

The three rather different situations described above are usually emphasized through a very popular illustration dealing with the static equilibrium of rigid body reported in Figure 1.1.

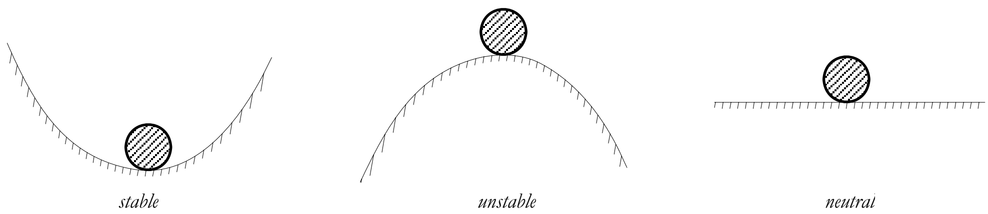

Figure 1.1: Three different equilibrium conditions of a ball in a gravity field

Since equilibrium problems can be (and usually are) solved in terms of balance between force acting on the (free) body, a *static approach* can be even pursuit not only for finding the equilibrium configuration of a system (namely, those in which acting and reacting forces definitely balance one another), but even for discussing the "quality" of such an equilibrium condition in terms of the three possible attributes listed and defined above.

Nevertheless, since such definition has been done in terms of "perturbations" and "oscillations" a *dynamic approach* to equilibrium seems needed to go in depth about its "quality" in the sense reported above. Equation of motions can be written for the system starting from its current equilibrium condition to ascertain whether it tends to come back to the original condition or evolves towards other more stable ones. Writing, solving and discussing dynamic equations usually results in more complicated mathematical procedures involving parameters non-needed by other ones (such as mass of body).

On the contrary, one more different approach can be followed for studying stability of systems and structures. In fact, the three images reported above basically connect the mentioned "nature" or "quality" of equilibrium with the "position" of the equilibrium point. The physical meaningfulness of the three situations depicted in Figure 1.1 can be increased by the fact that in a gravity field the functional Potential Energy U. Consequently, a different way of facing the problem of stability of structures can be based on an *energy approach* for which a conceptual correspondence can be stated with Figure 1.1, as will be explained within the next paragraphs.

The basic definitions given for the "quality" of equilibrium configuration with reference to the simple free body depicted in Figure 1.1 need to be generalized to structural systems usually made of flexible member connected one another for facing the loads and action applied on them. Nevertheless, a hopefully meaningful application of the three mentioned approaches will be carried out in following sections of the present chapter on simpler systems made of rigid bodies connected one another by a finite number of flexible elements (basically linear springs) with the aim of providing the readers with the basic insights on stability of structural systems. These structures are usually named "discrete system" and so will be done in the following of this text.

Euler definition of stability will be firstly referred to discrete structures pointing out the condition of possible alternative equilibrium configuration available for the system (affected by no imperfections and in equilibrium in its reference position) when (axial) loads achieve a certain "critical" value: this possible availability of more than one configuration is currently named as "bifurcation" of equilibrium. Moreover, imperfections do play a relevant role in the so called "equilibrium path" (namely the relationship between the current value of loads and the corresponding equilibrium configuration of the system) and their influence will be even analyzed in this chapter with reference to the mentioned discrete systems.

Finally, bifurcation, instability onset (namely "buckling") and the way in witch initial imperfections of the system affect them will be analyzed in the light of the Koiter's theorems which are a very powerful tool for understanding the "quality" of a given equilibrium position.

1.2 Euler definition of stability

Equilibrium of bodies or structures can be studied within the framework of various hypotheses. In particular, a relevant class of equilibrium problems is usually named Euler problems and is based upon the two following hypotheses:

- small displacements of the structures starting from a reference equilibrium configuration;
- perfect or "ideal" systems, namely absence of geometric and mechanical imperfections (accidental eccentricity, lack of straightness or verticality, and so on).

When these two hypotheses are contemporarily considered in equilibrium problems the Euler definition of stability can e investigated with the implications that will be pointed out in the following with reference to simple systems and according to different methodological approaches.

1.3 Discrete systems

In the present section, three possible approaches for studying the stability of equilibrium will be outlined; application to simple discrete systems will point out the conceptual and operational differences among them. Two basic reasons justify the choice of discrete systems rather than the most common continuous ones:

- first of all, since the focus in the following discussion is on meaning of different approaches, examples characterized by simple calculations (such as needed and sufficient for discrete systems) have been preferred;
- analysis of continuous structures is usually carried out through discretization techniques (Finite Element Method, among the others); consequently the mathematical formulation of the equilibrium and stability problems for continuous structures could be not so different with respect to discrete ones.

1.3.1 Static approach

The first approach for stating equilibrium problems is based on a static approach in which the sum of actions and reactions (in terms of either forces or moments) are imposed to be zero. Consequently, equilibrium equation can be directly imposed on a deformed configuration in the neighbours of the reference one considering the effect of displacements of the forces applied of the structure. For instance, one can consider the simple discrete system made out of a rigid bar whose length is L, connected to the soil by a hinge and a rotational spring k and loaded of the opposite end by an axial load P (Figure 1.2).

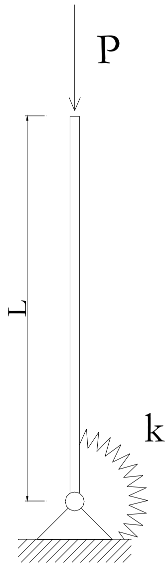

Figure 1.2: Discrete system #1

According to the Euler hypotheses listed in paragraph 1.2 the bar is straight and the load is perfectly centred and vertical in position. The static approach can be now pursued by considering a deformed configuration for the system which is described by the only kinematical parameter θ (Figure 1.3).

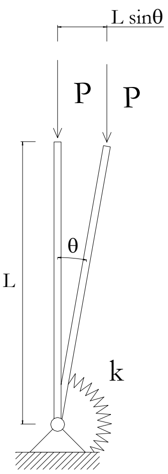

Figure 1.3: Discrete system #1: deformed configuration for applying the static approach

A simple equilibrium condition can be stated in terms of moments around the base hinge; only two contributions participate play a role in the equilibrium of the above system: the axial load applied on the deformed configuration and the rotational spring, was moment is proportional to the actual rotation θ:

$$PL\sin\theta - k\theta = 0 \quad . \tag{1.1}$$

If the first of the two Euler hypotheses applies (namely, small displacements), then the value of the sine can be approximated by its argument and equation (1.1) can be written as follows:

$$\left[PL - k\right]\cdot\theta = 0 \quad . \tag{1.2}$$

The above equation points out that two possible solutions (and two corresponding equilibrium configurations) can be found for the system:

- the trivial configuration, described by the solution $\theta = 0$ which firstly satisfy the above equilibrium equation;
- a bifurcated solution obtained as the first factor of equation (1.2) vanishes:

$$PL - k = 0 \Rightarrow P = \frac{k}{L} = P_E \;; \tag{1.3}$$

such a condition occurs as the external load attains the value P_E, which is a bifurcation load in the sense that under P_E the system can switch out from its (trivial) configuration toward a bifurcated one.

Static approach carried out under the mentioned Euler hypotheses, leads to describe the equilibrium paths as a bifurcation phenomenon which can have place for a precise value of the external load P equal to P_E. Figure 1.4 show the equilibrium path, namely the couples (θ, P) obtained by solving equation (1.2).

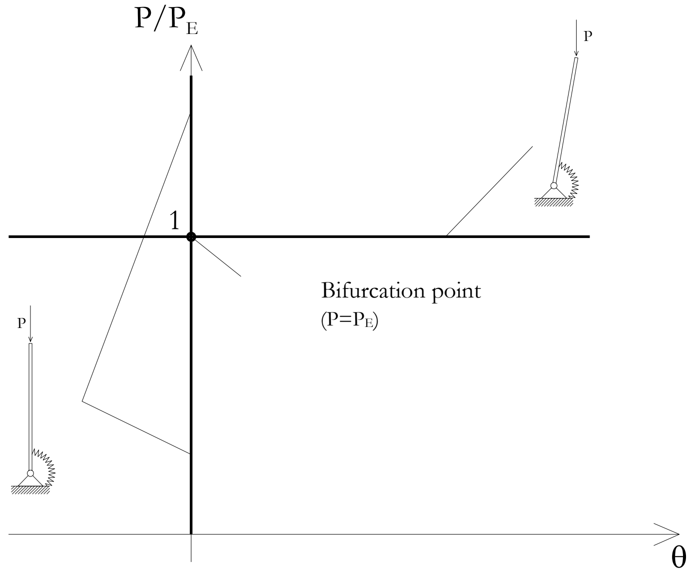

Figure 1.4: Discrete system #1: bifurcation phenomenon derived under the Euler assumptions

Although the static approach lead to a rather simple way for evaluating all the possible equilibrium configurations of the system, nothing can be deduced for qualifying the single branches of the equilibrium path in terms of stability. Physical sense could suggest that the vertical branch would be of stable form the origin to $P<P_E$ and unstable above that value, but no quantitative evaluation can be done on that issue. Moreover, nothing can be understood for the linear branch, where the key hypothesis of small displacement is not always verified.

1.3.2 Dynamic approach

If one keeps in mind the rather intuitive definitions of stability corresponding to the basic ideas depicted in Figure 1.1, dynamic approach for stability would be a somehow "natural" framework for studying equilibrium and stability of bodies and structures. Indeed, the idea of stable equilibrium is associated to small perturbations resulting in oscillation of the body around its initial configuration; on the contrary, a diverging motion stems by even small perturbations if the initial equilibrium configuration is unstable in nature. The words "oscillation" and "motion" are strictly related to the dynamic behaviour of the system after its perturbation. Consequently, if one looks after the same simple system mentioned above, whose configurations are described by only one kinematical parameter, the equation of motion stemming from the initial perturbation leading the system in a different position described by $\theta=\theta_0>0$ can be easily obtained through the well-known D'Alembert Principle introducing the mass *m* of the system:

$$F - m\ddot{z} = 0 \;, \tag{1.4}$$

that can be written as follows in terms of moments and angular frequency (rather than forces F and linear acceleration $\ddot{z}$ because of the circular nature of motion):

$$M - I\ddot{\theta} = 0 \ . \tag{1.5}$$

In the last equation M is the sum of moments of the external forces with respect to the hinged end of the bar:

$$M = PL\sin\theta - k\theta \ , \tag{1.6}$$

while the second moment I of the bar with respect to the hinged end can be evaluated as follows:

$$I = \int_0^L \mu x^2 dx = \frac{\mu L^3}{3} \ , \tag{1.7}$$

being μ the linear mass density of the system (Figure 1.5).

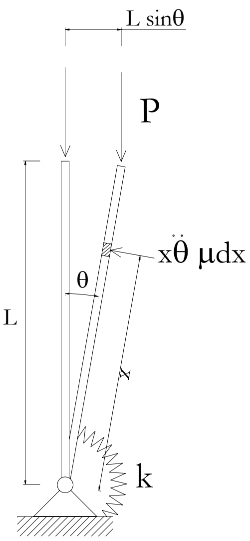

Figure 1.5: Discrete system #1: Dynamic approach

Equation (1.5) can be easily simplified by introducing the lat two expressions obtaining the following equation of motion:

$$PL\sin\theta - k\theta - \frac{\mu L^3}{3}\cdot\ddot{\theta} = 0 \ . \tag{1.8}$$

According to the first of the two Euler hypotheses, the sine function can be well approximated by its argument and the following expression can be obtained for the equation of motion:

$$\ddot{\theta} - \frac{3}{\mu L^3}\left[PL - k\right]\theta = 0 \ , \tag{1.9}$$

which is a linear second-order differential equation whose solution depends upon the sign of the following term:

$$\omega^2 = \frac{3}{\mu L^3}\left[PL - k\right] \ . \tag{1.10}$$

The two cases described below can occur:

- $\frac{3}{\mu L^3}\left[PL - k\right] < 0$, namely $P < \frac{k}{L} = P_E$: in this case the equation of motion takes the following form:

$$\ddot{\theta} + \left|\omega^2\right|\theta = 0 \ , \tag{1.11}$$

whose general solution is:

$$\theta(t) = A\sin|\omega|t + B\cos|\omega|t \ , \tag{1.12}$$

and the constant terms A and B could be found imposing the suitable initial conditions in terms of displacement $\theta(0)$ and velocity $\dot{\theta}(0)$. For whatever values of such initial condition the motion consists in oscillations of the body around the initial configuration due to the oscillatory nature of trigonometric functions: consequently, if $P<P_E$ the equilibrium in stable;

- $\frac{3}{\mu L^3}[PL-k]>0$, namely $P>\frac{k}{L}=P_E$: in this case the equation of motion takes the following form:

$$\ddot{\theta}-\omega^2\theta=0 \ , \tag{1.13}$$

whose general solution is:

$$\theta(t)=Ae^{\omega t}+Be^{-\omega t} \ , \tag{1.14}$$

and the constant terms A and B could be found imposing the suitable initial conditions in terms of displacement $\theta(0)$ and velocity $\dot{\theta}(0)$. Nevertheless, the nature of motion is always diverging since the positive exponential of equation (1.14) is a strictly increasing function which describe the motion of the body which abandon the initial equilibrium position toward another configuration: consequently, if $P>P_E$ the equilibrium in unstable.

Finally, besides its procedural difficulties, dynamical approach is the closest one with respect to the intuitive definition of equilibrium stability represented in Figure 1.1 for a rigid ball within the gravitational field. Consequently, further information about the quality of equilibrium has been drawn for the two vertical branches of the equilibrium path represented in Figure 1.4., showing that the lower part ($P<P_E$) is stable in nature, while the upper one ($P>P_E$) is unstable. On the contrary, nothing can be said about the nature of the horizontal post-bifurcation branch: large displacements occur after bifurcation and Euler can only provide the bifurcation point, but is not fit for describing post-buckling behaviour.

1.3.3 Energy approach

A third possible approach can be put in place for facing the problem of equilibrium and stability of bodies and structures. This approach can be once more introduced starting from the sample problem of the rigid ball within the gravitational field, whose equilibrium configurations are achieved in points of minimum or maximum height. Indeed, only in those points the work made by the (vertical) gravitational force for all the possible displacements (namely, virtual displacements) is zero. This condition can be generalized to the case of elastic structures through the Principle of Minimum of Total Potential Energy Π which can be defined as the sum of the elastic deformation energy E and the (opposite of) the work W made out by the external loads:

$$\Pi=E-W \ . \tag{1.15}$$

The functional Π is defined in terms of the displacements needed for describing the deformed shape of the given elastic system. In particular, a simple analytical expression can be found for the discrete system represented in Figure 1.3:

$$\Pi(\theta)=\frac{1}{2}k\cdot\theta^2-PL\cdot(1-\cos\theta) \ , \tag{1.16}$$

and the equilibrium configurations can be determined through the mentioned principle as points of stationarity (namely, values of the parameters in which the gradient of Π is zero) as follows:

$$\frac{d\Pi}{d\theta}=0\Rightarrow k\cdot\theta-PL\cdot\sin\theta=0 \ . \tag{1.17}$$

The above equation can be solved under the usual small displacement hypothesis considered in the Euler formulation of equilibrium and stability:

$$[k-PL]\cdot\theta=0 \ . \tag{1.18}$$

and the bifurcation phenomenon occur as $P=P_E$.

The nature of equilibrium can be deduced by studying the sign of the second derivative (namely, the Hessian matrix, in the case of multi-degree of freedom systems):

- equilibrium is stable if the function Π has a local minimum point, namely, in the cases of single degree of freedom systems, the second derivative is positive;
- equilibrium is unstable if the function Π has a local maximum and, consequently, the second derivative is negative.

The second derivative of Π can be easily determined as follows:

$$\frac{d^2\Pi}{dz^2} = k - PL\cos\theta \approx k - PL \ , \tag{1.19}$$

and its sign can be easily derived:

- $\frac{d^2\Pi}{dz^2} > 0$ as $k > PL \Rightarrow P<P_E$, meaning that equilibrium is stable in the lower part of the vertical branch;
- $\frac{d^2\Pi}{dz^2} < 0$ as $k < PL \Rightarrow P>P_E$, meaning that equilibrium is unstable in the upper part of the vertical branch.

Finally, the same conclusions derived through the dynamic approach can be achieved by the energy-based approach for determining the equilibrium positions of the systems and its stability. No further insights have been derived about the post-bifurcation branch as a result of the limitation imposed by the small displacement hypothesis that will be finally removed in the next section for understanding the post-buckling behaviour in terms of equilibrium path and its stability conditions.

1.4 Post-buckling behaviour

Euler problems are based on the two mentioned assumptions (small displacements and perfect systems) and under those hypotheses the equilibrium of structures and bodies can be found for a basic "trivial" configuration, but under given load values the system configuration can switch toward an adjacent one. Since large displacements naturally originate after bifurcation, the study of post-buckling behaviour of structures can be pursued by considering large displacements. Consequently, the above calculation can be made anew by without simplifying the expressions obtained through one of the three approaches described above. In particular, as a result of its conceptual comprehensiveness and practical simplicity, the energy-based approach will be pursued in the following. The equilibrium condition for the above mentioned system is described by equation (1.17) which can be solved as follows with respect to the external load P:

$$\frac{PL}{k} = \frac{\theta}{\sin\theta} \ , \tag{1.20}$$

and, introducing the definition of the Euler load P_E given by (1.3), the following equation can be obtained:

$$\frac{P}{P_E} = \frac{\theta}{\sin\theta} \ . \tag{1.21}$$

Equations (1.20) and (1.21) are represented in Figure 1.6 where a curved path, stemming out form the bifurcation point, is ruled by the relationship reported at the right member of both equations.

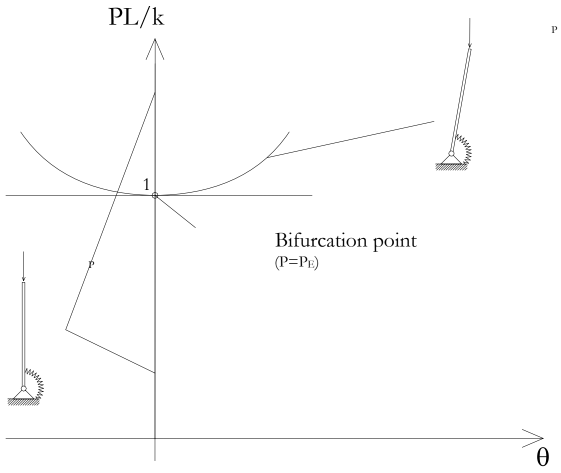

Figure 1.6: Discrete system #1: Post-Buckling behaviour

The quality of equilibrium throughout this curved path reported within the (θ, P/P_E) plane can be discussed by looking after the second derivative of the Total Potential Energy Π already evaluated and reported in equation (1.19) where the expressions without small displacement simplification is of interest for the post-buckling behaviour:

$$\frac{d^2\Pi}{dz^2} = k - PL\cos\theta \quad . \tag{1.22}$$

The sign of the second derivative of Π throughout the post-buckling path can be deduced by introducing equation (1.20) in (1.22) and obtaining the following condition:

$$\frac{d^2\Pi}{dz^2} > 0 \Leftrightarrow 1 - \frac{\theta}{\tan\theta} > 0 \quad , \tag{1.23}$$

which is true for every value of θ belonging to the range [0, $\pi/2$). Consequently the post-buckling behaviour is stable.

Nevertheless, a variety of possible behaviours can occur after bifurcation. To have an example of a significantly different one, the system represented in Figure 1.7 can be examined.

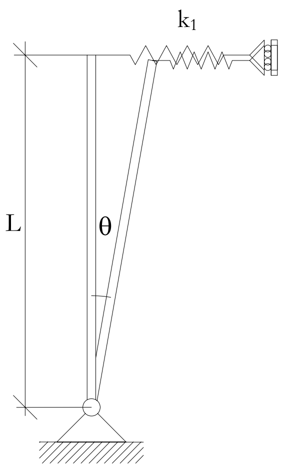

Figure 1.7: Discrete system #2: Trivial and adjacent configurations

According to the energy-based approach, the following expression for the Total Potential Energy Π can be written as follows:

$$\Pi(\theta) = \frac{1}{2}k_1 \cdot \left[L\sin\theta\right]^2 - PL \cdot \left(1-\cos\theta\right) \quad , \tag{1.24}$$

and its first derivative can be evaluated as follows

$$\frac{d\Pi}{d\theta} = k_1 L^2 \cdot \sin\theta\cos\theta - PL \cdot \sin\theta = 0 \ . \qquad (1.25)$$

If both the basic assumptions of the Euler approach apply, the first-order approximation can be adopted for the above equation whose final expression is reported below

$$\left[k_1 L - P\right] \cdot \theta = 0 \ , \qquad (1.26)$$

which defines the value of the Euler load $P_E = k_1 L$ resulting in a bifurcation in equilibrium path of the system. Figure 1.8 shows that the second system analyzed under the Euler assumption behaves like the first one, since the equilibrium path consists of a vertical branch corresponding to the trivial configuration (θ=0) and an horizontal one (P/P_E=1) representing the bifurcation condition attained for the Euler critical value P_E.

Post-buckling behaviour is even of interest for this system, since significant differences with respect to the first one could be pointed out, even if a formal equivalence between the two systems has been deduced under the Euler assumptions. Equation (1.25) describes the post-buckling behaviour in large displacements and can be easily solved with respect to the external load P and introducing the above definition of the Euler load P_E:

$$\frac{P}{k_1 L} = \frac{P}{P_E} = \cos\theta \ , \qquad (1.27)$$

Equation (1.27) describing the equilibrium path is even represented in Figure 1.8 which points out as post-buckling is a decreasing branch stemming out from the bifurcation point.

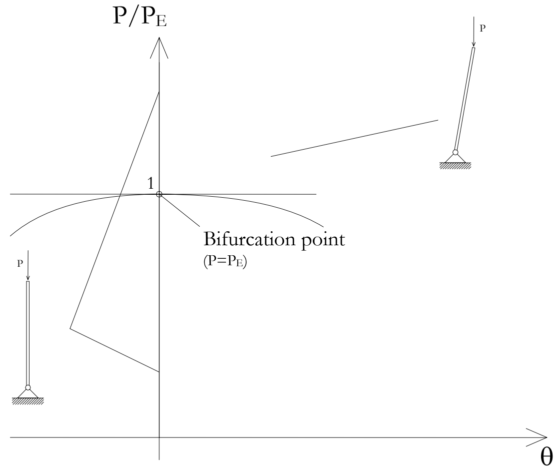

Figure 1.8: Discrete system #2: Bifurcation and Post-Buckling behaviour

The second derivative of P as to be studied for understanding whether the equilibrium is stable or not:

$$\frac{d^2\Pi}{d\theta^2} = k_1 L^2 \cdot \cos^2\theta - k_1 L^2 \cdot \sin^2\theta - PL \cdot \cos\theta \ ; \qquad (1.28)$$

the sign of the second derivative can be studied after dividing by $k_1 L^2$ and introducing the equation (1.27) of the post-buckling branch:

$$\frac{d^2\Pi}{d\theta^2} > 0 \Leftrightarrow \cos^2\theta - 1 > 0 \ ; \qquad (1.29)$$

which is not true for any value of θ within the range [0, π/2). Consequently, the second system is subjected to an unstable equilibrium bifurcation.

Finally, post-buckling analysis points out the key aspects related to the stability or instability of the equilibrium path after bifurcation that can be even foreseen by the simple Euler theory was assumptions are not restrictive since perfect systems have small displacements.

1.5 Imperfection sensitivities

The previous chapter has been devoted to study the post-buckling behaviour after removing the small displacement hypothesis that is one of the two basic assumptions within the Euler problems. The second one, dealing with the idea that the bodies or structures are not affected by any imperfection (i.e. lack of verticality or accidental eccentricity), will be removed herein with the basic aim of understanding the behaviour of systems which are closer to the "real" ones which are usually affected by imperfections due to the producing or building process.

Consequently, an initial eccentricity can be introduced for the load P in the first system considered above. The analytical expression of the Total Potential Energy Π for this system can be written as follows:

$$\Pi(\theta) = \frac{1}{2} k \cdot \theta^2 - P\left[L \cdot (1-\cos\theta) + e \cdot \cos\theta\right] \ , \tag{1.30}$$

for taking into account the contribution of eccentricity e to the vertical displacement of the loaded tip. The first derivative of Π can be easily obtained and imposed equal to zero:

$$\frac{d\Pi}{d\theta} = k \cdot \theta - P\left[L \cdot \sin\theta + e \cdot \cos\theta\right] = 0 \ , \tag{1.31}$$

whose solution in terms of P is reported below:

$$\frac{PL}{k} = \frac{\theta}{\sin\theta + e/L \cdot \cos\theta} \ . \tag{1.32}$$

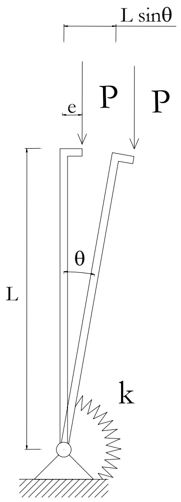

Figure 1.9: Discrete system #1: initial eccentricity as possible imperfection

Since the Euler critical load $P_E = k/L$, the above expression can be reported in terms of the P/P_E ratio as follows:

$$\frac{P}{P_E} = \frac{\theta}{\sin\theta + e/L \cdot \cos\theta} \ . \tag{1.33}$$

The above formula can even reproduce the case of "perfect" system studied in chapter 1.3; in fact, in the case of $e/L = 0$ the equation (1.33) reduces to (1.21). In the general case, namely for $e/L > 0$, equation (1.33) describes the family of curves represented in Figure 1.10 which are as far from the response of the perfect system as the value of the non-dimensional eccentricity e/L is great. Moreover, lateral displacements even occur for values of load P lesser than the theoretical P_E, but initial imperfection does not affect the ultimate load of the system whose load path gets closer and closer to the post-buckling behaviour of the perfect system.

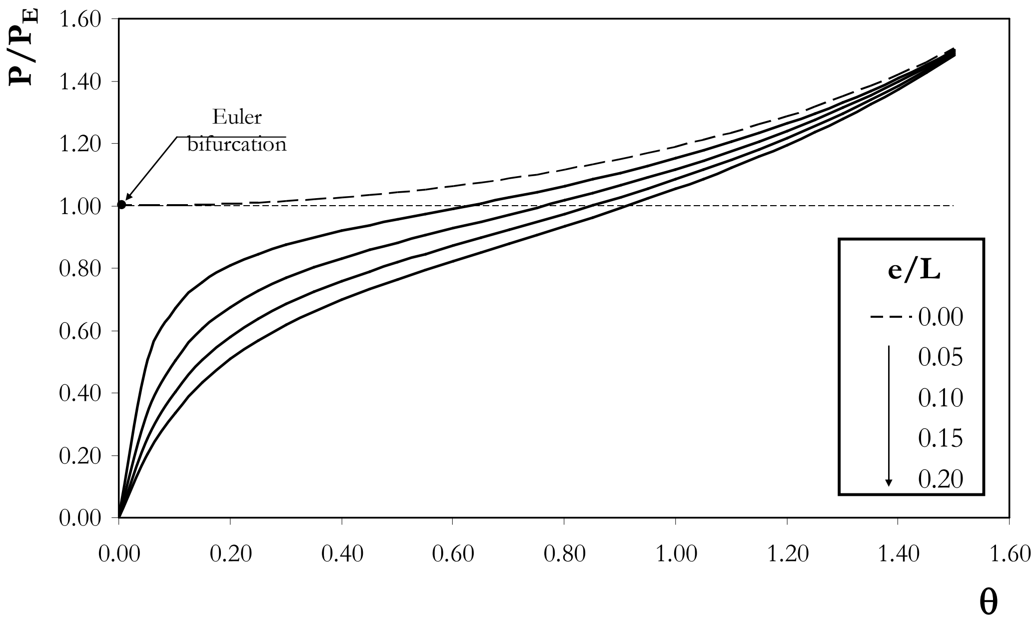

Figure 1.10: Discrete system #1: the role of imperfection on equilibrium paths

Although no further analytical developments will e proposed herein for the sake of brevity, it would be easy to demonstrate that the equilibrium for the system is stable in nature since its second derivative is strictly positive throughout the entire equilibrium path.

Finally, the system #1 (and, by extension all those systems with stable post-buckling behaviour) are not imperfection sensitive in terms of ultimate load, but have only larger lateral displacement as a result of those imperfection.

The second system can be now studied considering that it is affected by an imperfection whose effect can be simulated by an initial out-of-verticality angle θ_0 (Figure 1.11).

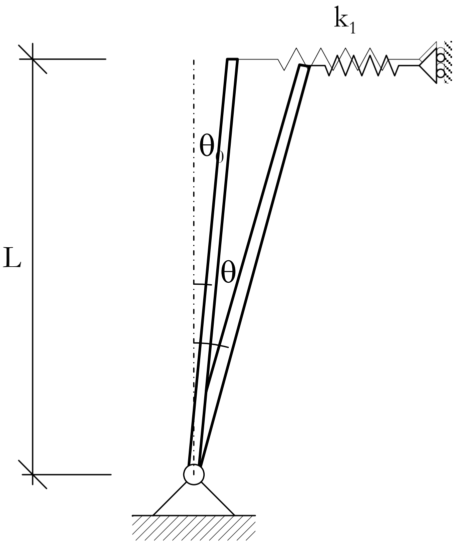

Figure 1.11: Discrete system #2: out-of-verticality as possible imperfection

Starting from that deviated configuration, the expression of the Total Potential Energy can be defined for $\theta \geq \theta_0$ and written as follows:

$$\Pi(\theta) = \frac{1}{2} k_1 \cdot \left[L\sin\theta - L\sin\theta_0 \right]^2 - PL \cdot \left(\cos\theta_0 - \cos\theta \right) \ , \tag{1.34}$$

and the corresponding equilibrium configuration of the system can be sought by solving the following equation:

$$\frac{d\Pi}{d\theta} = 0 \Rightarrow k_1 L^2 \cdot \left[\sin\theta - \sin\theta_0\right]\cos\theta - PL \cdot \sin\theta = 0 \ . \tag{1.35}$$

Simplifying the above equation and introducing the definition of Euler load P_E stated by equation (1.26), the following relationship can be obtained for the equilibrium path:

$$\frac{P}{P_E} = \frac{\left(\sin\theta - \sin\theta_0\right)}{\sin\theta} \cdot \cos\theta \ , \tag{1.36}$$

and in the case of "ideal" system ($\theta_0 = 0$) equation (1.36) reduces to (1.27). Figure 1.12 plots that relationship as a dashed line; the continuous lines reproducing the behaviour of the system with imperfection are hugely affected by imperfections in terms of both lateral displacements in the ascending branch and ultimate load which is usually as smaller than P_E as the imperfection amplitude θ_0.

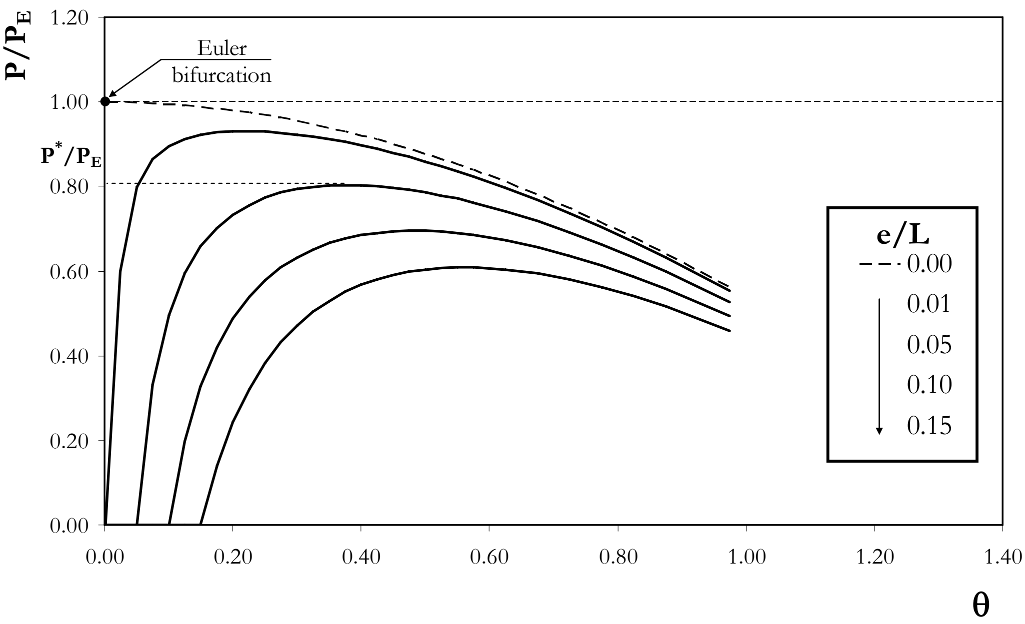

Figure 1.12: Discrete system #2: the role of imperfection on equilibrium paths

Consequently, two achievements have been drawn by the above two systems that can be assumed as paradigms of two classes of problems:

- in the first one, stable (and symmetric) behaviour occurs after bifurcation imperfection only affects the lateral flexibility of the system;
- for the second one unstable bifurcation occurs and, when imperfections are taken into account, both lateral stiffness and ultimate load are hugely affected.

Finally, the intimate relationship between post-buckling behaviour and imperfection sensitiveness will be emphasized in the next section as one of the key aspects of a more general theory.

1.6 Koiter's theory

1.7 Applications

1.7.1 Worked examples

1.7.2 Unworked examples