5. Buckling analysis of plane structures: simplified methods

The present chapter addresses the basic concepts of stability analysis of (plane) frames; definition of structures is firstly considered with the aim of recognizing various possible structural behaviours in terms of stability. Secondly, a practical method for evaluating the critical multiplier (namely, the value of normal and/or horizontal action resulting in global buckling of structures) will be formulated and commented. Simplified methods of analysis which can used for taking into account the possible effects of axial loads on lateral behaviour of steel structures will be finally described. EC3 [14] provisions will be assumed and reported about the above topics.

The above concepts are applied in worked examples, while unworked ones are left to the reader for further study.

5.1 Definition and classification of structures

Different possible classifications can be invoked and adopted for structures considering the various aspects of their behaviour. In particular, if one looks after the behaviour in terms of response to lateral actions the following classification can be introduced for the framed structures:

- *non-sway structures*, in which the influence of vertical loads on lateral displacements (and the other related quantities such as ending and shear stresses in beam-columns) in negligible;
- *sway structures*, in which axial load contribution to lateral displacements is not negligible and stresses in structural members cannot be easily evaluated by considering the axial loads on the undeformed configuration of the frame structure.

The small displacements hypotheses, usually assumed in structural analysis of frames, is carried out by considering all the action applied on the undeformed configuration of the structure, possibly affected by some kinds of imperfections (typically, the lack of verticality) as provided by the relevant Codes of Standards. The analysis carried out on the undeformed shape is usually called *first order analysis* because the strain measures are simply derived as first order derivatives of the displacements and the corresponding stresses neglect the change in eccentricity due to the actual configuration of the frame. As far as lateral displacements increases, the effects of the eccentricity of vertical loads on the actual value of stresses is more and more relevant and a *second order analysis* considering loads on the deformed shape is needed for obtaining a careful estimation of the final state of stress of the structure as a whole.

Since loads on structures are always applied on their actual deformed configuration, second order analysis always results in a more refined simulation of the structural response. Nevertheless, in non-sway structures the difference of the results stemming out by first order analysis are reasonably close to the ones derived by second order analysis. In particular, if the difference the lateral displacements Δ_1 derived by first order analysis and the corresponding total displacement Δ_{tot} which takes into account of the increase in displacements due to the increased eccentricity in vertical load is within the 10% of Δ_1, and then the corresponding effects can be neglected.

The condition assumed by Code of Standards for dividing non-sway structures by the sway ones is often given in terms of the critical multiplier α_{cr}, namely is the factor by which the design loading would have to be increased to cause elastic instability in a global mode. Consequently, if F_{Sd} is the design load on structure and F_{cr} is the corresponding values of forces resulting in global instability of the same structure, the following limitation has to be checked for ascertain that the given structure belongs to the class of non-sway-frames:

$$\alpha_{cr} = \frac{F_{cr}}{F_{Sd}} \leq 0.10 \ . \tag{5.1}$$

The above limitation is drawn out by EC3 [14] for elastic analysis of structures; a larger limitation for the critical multiplier of non-sway structures is given when plastic analysis is performed:

$$\alpha_{cr} = \frac{F_{cr}}{F_{Sd}} \leq 0.15 \ . \tag{5.2}$$

because structural behaviour may be significantly influenced by non linear material properties in the ultimate limit state (e.g. where a frame forms plastic hinges with moment redistributions or where significant non linear deformations from semi-rigid joints occur).

For reasons that will be hopefully clearer in paragraph 5.3, the limitation given in (5.1) is conceptually and quantitatively equivalent to the 10% limitation on displacement increase due to second order (otherwise called P-Δ) effects.

5.2 Critical multiplier of plane frames: Horne method

According to the content of the previous section, the classification of structures in non-sway and sway basically reduced to the issue of quantifying the critical load multiplier α_{cr} or, equivalently, determining the critical load F_{cr} for the framed structure. Non-linear analysis carried out by refined numerical methods, such as the Finite Element Method, are currently available for carrying out the so-called buckling analysis of structures, evaluating the external forces resulting in indefinite lateral displacements for the given structures. Mention, formulation and comments about these advanced methods can be found, for instance, in [7] and other works on computational mechanics.

In the present section, a simplified method which can be easily utilized for ordinary design purposes is the so-called Horne Method [5], which is also adopted (under certain limitations) by EC3 [14].

The following expression can be evaluated at each storey of the given frames structures:

$$\alpha_{cr,k} = \frac{H_{Sd,k} \cdot h_k}{V_{Sd,k} \cdot \delta_{H,Sd,k}} \ , \tag{5.3}$$

with the following meaning of the symbols:

- $H_{Sd,k}$ the total (horizontal) shear force at the k-th storey;
- $V_{Sd,k}$ the total vertical load at the k-th storey;
- h_k the storey height at the k-th storey;
- $\delta_{H,Sd,k}$ in interstorey drift displacement at the k-th storey.

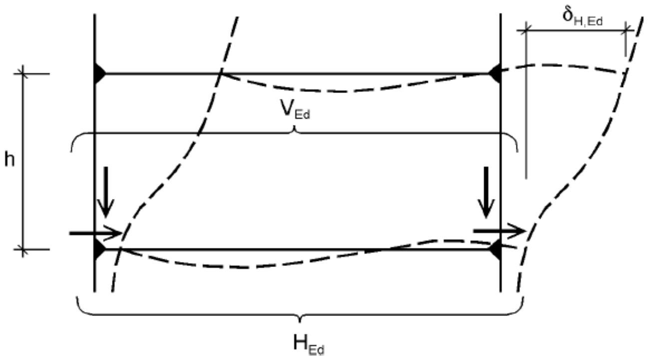

Figure 5.1: Simply supported beam with torsional end restraints.

The critical load can be defined as the minimum value assumed by the quantity $a_{cr,k}$ at the various storeys:

$$\alpha_{cr} = \min_k \left\{ \alpha_{cr,k} \right\} \ . \tag{5.4}$$

If only vertical loads are applied on the structure, the following alternative expression can be assumed for the critical multiplier $\alpha_{\chi\rho}$:

$$\alpha_{cr} = 0.9 \cdot \min_k \left\{ \frac{h_k}{\delta_{H,Sd,k}} \right\} \ . \tag{5.5}$$

Finally, the simplified method described above basically covers all the cases of rectangular frames and can be applied by simply carrying out a linear first-order analysis of the structure without the need for more advanced analysis tools.

5.3 Second-order analysis of sway structures: simplified methods

The methods reported in the previous chapter allows the designer to establish whether the given frame can be regarded as non-sway or sway structure by checking relationship (5.1) after evaluating the critical multiplier by equation (5.4) or (5.5).

Consequently, the present section is devoted to the case of sway structures which cannot be easily analyzed with reference to the undeformed shape. General numerical procedures addressing the issues of the so-called geometric non-linearity are available nowadays, but the present paragraph will only focus on a simple procedure called the *N-Δ method*.

Let us consider the single storey rectangular frame represented in Figure 5.2: it is a shear-type structure, in which no nodal rotation occur since the horizontal beam is infinitely stiff with respect to the columns. Let K_t be its lateral stiffness and, under the shear type hypothesis, the following expression can be assumed:

$$K_t = 4 \cdot \frac{12EI}{H^3} = \frac{48EI}{H^3} \quad . \tag{5.6}$$

Consequently, the first-order lateral displacement Δ_1 can be easily determined as a function of the lateral force:

$$\Delta_1 = \frac{F}{K_t} \quad . \tag{5.7}$$

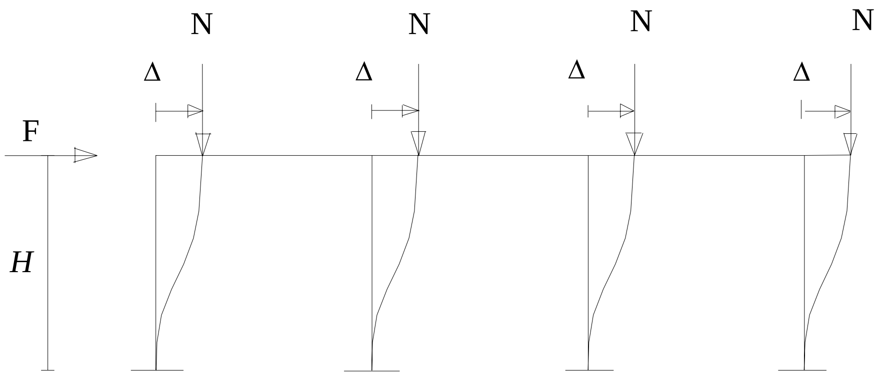

Figure 5.2: Single frame rectangular structure.

Due to the onset of the lateral displacement Δ_1, the total axial loads *4N* are eccentrically applied and a total moment $4N\Delta_1$ and, by equilibrium, a fictitious horizontal force ΔF_1 can be introduced as follows:

$$\Delta F_1 = \frac{4N}{H} \cdot \Delta_1 \quad . \tag{5.8}$$

resulting in a further contribution Δ_2 to horizontal displacements

$$\Delta_2 = \frac{\Delta F_1}{K_t} = \frac{4N}{HK_t} \cdot \Delta_1 \quad . \tag{5.9}$$

The above formula summarizes the so-called N-Δ method and the total displacement Δ_{tot} can be approximately evaluated as follows:

$$\Delta_{tot} \approx \Delta_1 + \Delta_2 \quad . \tag{5.10}$$

Nevertheless, the displacement Δ_2 increases the vertical load eccentricities and a further lateral force can be introduced for simulating its effect according to the equilibrium equation (5.8):

$$\Delta F_2 = \frac{4N}{H} \cdot \Delta_2 = \left(\frac{4N}{HK_t}\right)^2 \cdot K_t \Delta_1 \ , \tag{5.11}$$

and the corresponding increase in terms of lateral displacements is:

$$\Delta_3 = \frac{\Delta F_2}{K_t} = \left(\frac{4N}{HK_t}\right)^2 \cdot \Delta_1 \ . \tag{5.12}$$

Consequently, on the bases of the above relationship, a recursive relationship can be written to evaluate the i-th contribution to lateral displacements according to the above method:

$$\Delta_i = \frac{\Delta F_{i-1}}{K_t} = \left(\frac{4N}{HK_t}\right)^{i-1} \cdot \Delta_1 \ . \tag{5.13}$$

Finally, the total displacement can be evaluated by summing all the partial contributions Δ_i obtained by considering the axial loads on the deformed shape of the frame:

$$\Delta_{tot} = \sum_{i=1}^{\infty} \Delta_i = \sum_{i=1}^{\infty} \left(\frac{4N}{HK_t}\right)^{i-1} \cdot \Delta_1 = \Delta_1 \left[1 + \left(\frac{4N}{HK_t}\right) + \left(\frac{4N}{HK_t}\right)^2 + \left(\frac{4N}{HK_t}\right)^3 + \ldots\right] . \tag{5.14}$$

The above formula, stemming out by the recursive application, represent a *geometric sequence* whose common ratio is $4N/HK_t$; the sum of this sequence can be derived by the following equation:

$$\Delta_{tot} = \Delta_1 \left[\frac{1}{1 - \dfrac{4N}{HK_t}}\right] . \tag{5.15}$$

A deep mechanical meaning can be also recognized for the term $4N/HK_t$ by applying the Horne Method described in the previous section for determining the critical multiplier for the structure represented in Figure 5.2:

$$\alpha_{cr} = \frac{H_{Sd} \cdot h}{V_{Sd} \cdot \delta_{H,Sd}} = \frac{F \cdot H}{4N \cdot \Delta_1} \ . \tag{5.16}$$

and introducing the relationship (5.7) between the lateral force F and the first-order displacement Δ_1 the following formula can be obtained:

$$\alpha_{cr} = \frac{K_t \cdot H}{4N} \Rightarrow \frac{1}{\alpha_{cr}} = \frac{4N}{K_t \cdot H} \ . \tag{5.17}$$

Finally, the second equality reported in equation (5.17) can be introduced in (5.15), obtaining a closed-form relationship between the total displacement Δ_{tot} and and its first order estimation Δ_1, involving the critical multiplier:

$$\Delta_{tot} = \Delta_1 \cdot \left[\frac{1}{1 - \dfrac{1}{\alpha_{cr}}}\right] . \tag{5.18}$$

EC3 [14] suggests the above formula for magnifying the first order effects (not only displacements, but even stresses and the other terms of the structural response) by the factor reported in equation (5.18). This procedure can be utilized in the following cases:

- single storey structures with $\alpha_{cr} \geq 3.0$, while for smaller values of the critical multiplier more refined analysis methods are required;
- multi-storey buildings, provided that all storeys have a similar distribution of vertical loads and distribution of horizontal loads and distribution of frame stiffness with respect to the applied storey shear forces.

Two final observations are now relevant:

1) the factor defined in equation (5.18) for framed structures has the same expression of the magnification factor introduced in paragraph 2.2 with the aim of evaluating the parameters of the beam-column response (in terms of displacements, stresses etc.) taking into account the effect of axial force on the deformed shape;

2) an explicit threshold has been mentioned in section 5.1 in terms of total-to-first-order displacement ratio for a structure to be classified as non-sway. In particular, second-order (N-Δ) were deemed negligible if

$$\frac{\Delta_{tot}}{\Delta_1} \le 1.10 \ ; \tag{5.19}$$

furthermore, the code limitation in terms of critical multiplier has been given in equation (5.1) and the substantial equivalence between these two definitions can be verified by evaluating equation (5.18) for the limit value $\alpha_{cr} = 10$:

$$\frac{\Delta_{tot}}{\Delta_1} = \left[\frac{1}{1 - \frac{1}{10}} \right] \approx 1.10 \ . \tag{5.20}$$

5.4 Worked examples

5.5 Unworked examples

The following unworked examples are proposed for verifying the understanding of the topics described within the present section:

1) with reference to the frame structure represented in Figure 5.2, design the column profiles with the aim of obtaining a non-sway structure according to EC3 provisions. Assume the following geometric and mechanical data:

- beam-column height	H	4.0 m;
- span length	B	6.0 m;
- axial load	N	300 kN;
- horizontal load	F	100 kN.

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